

# A new equivalence relation to classify the fuzzy subgroups of finite groups

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## Abstract

In this paper a new equivalence relation  $\approx$  to classify the fuzzy subgroups of finite groups is introduced and studied. This generalizes the equivalence relation  $\sim$  defined on the lattice of fuzzy subgroups of a finite group that has been used in our previous papers (see e.g. [16,24]). Explicit formulas for the number of distinct fuzzy subgroups with respect to  $\approx$  are obtained in some particular cases.

*Key words:* equivalence relations, fuzzy subgroups, chains of subgroups, groups of automorphisms, group actions, fixed points, cyclic groups, elementary abelian  $p$ -groups, dihedral groups, symmetric groups.

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## 1 Introduction

One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite group. This topic has enjoyed a rapid development in the last few years. Several papers have treated the particular case of finite abelian groups. Thus, in [8] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [9], [10], [11] and [27] deal with this number for cyclic groups of order  $p^n q^m$  ( $p, q$  primes). Recall here the paper [24] (see also [23]), where a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups: cyclic groups and elementary abelian  $p$ -groups. The explicit formula obtained for the first class leads in [16] to an important combinatorial result: a precise expression of the well-known central Delannoy numbers in an arbitrary dimension. Next, the study has been extended to some remarkable classes of nonabelian groups: dihedral groups, symmetric groups, finite  $p$ -groups having a cyclic maximal subgroup and hamiltonian groups (see [18], [20], [21] and [25], respectively). The same problems were also investigated for the fuzzy normal subgroups of a finite group (see [22]).

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Note that in all our papers mentioned above the fuzzy (normal) subgroups of finite groups have been classified up to the same natural equivalence relation  $\sim$  defined on the fuzzy (normal) subgroup lattices. This extends the equivalence relation used in Murali's papers [7]-[11] and gives a powerful connection between the fuzzy subgroups and certain chains of subgroups of finite groups. Recall here the technique initiated in [2] (see also [28]) to derive fuzzy theorems from their crisp versions. Some other different approaches to classify the fuzzy subgroups can be found in [4] and [5].

In the present paper we will treat the problem of classifying the fuzzy subgroups of a finite group  $G$  by using a new equivalence relation  $\approx$  on the lattice  $FL(G)$  of all fuzzy subgroups of  $G$ . This is more general as  $\sim$ , excepting the case when  $G$  is cyclic (for which we will prove that  $\approx = \sim$ ). On the other hand, its definition has a consistent group theoretical foundation, by involving the knowledge of the automorphism group associated to  $G$ . In order to count the distinct equivalence classes relative to  $\approx$ , we shall use an interesting result of combinatorial group theory: the Burnside's lemma (see [13] or [26]). Our method will be exemplified for several remarkable classes of finite groups. Also, we will compare the explicit formulas for the numbers of distinct fuzzy subgroups with respect to  $\approx$  with the similar ones obtained in the case of  $\sim$ . Our approach is motivated by the realization that in a theoretical study of fuzzy groups, fuzzy subgroups are distinguished by their level subgroups and not by their images in  $[0, 1]$ . Consequently, the study of some equivalence relations between the chains of level subgroups of fuzzy groups is very important. It can also lead to other significant results which are similar with the analogous results in classical group theory.

The paper is organized as follows. In Section 2 we present some preliminary results on the fuzzy subgroups and the group actions of a finite group  $G$ . Section 3 deals with a detailed description of the new equivalence relation  $\approx$  defined on  $FL(G)$  and of the technique that will be used to classify the fuzzy subgroups of  $G$ . These are counted in Section 4 for the following classes of finite groups: cyclic groups, elementary abelian  $p$ -groups, dihedral groups and symmetric groups. In the final section several conclusions and further research directions are indicated.

Most of our notation is standard and will usually not be repeated here. Basic notions and results on lattices, groups and fuzzy groups can be found in [1], [14] and [3] (see also [6]), respectively. For subgroup lattice concepts we refer the reader to [12] and [15].

## 2 Preliminaries

Let  $G$  be a group,  $\mathcal{F}(G)$  be the collection of all fuzzy subsets of  $G$  and  $FL(G)$  be the lattice of fuzzy subgroups of  $G$  (see e.g. [6]). The fuzzy (normal) sub-

groups of  $G$  can be classified up to some natural equivalence relations on  $\mathcal{F}(G)$ . One of them (used in [16]-[25], as well as in [27]) is defined by

$$\mu \sim \eta \text{ iff } (\mu(x) > \mu(y) \iff \eta(x) > \eta(y), \text{ for all } x, y \in G)$$

and two fuzzy (normal) subgroups  $\mu, \eta$  of  $G$  are said to be *distinct* if  $\mu \not\sim \eta$ . This equivalence relation generalizes that used in Murali's papers [7]-[11]. Also, it can be connected to the concept of level subgroup. In this way, suppose that the group  $G$  is finite and let  $\mu : G \rightarrow [0, 1]$  be a fuzzy (normal) subgroup of  $G$ . Put  $\mu(G) = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$  and assume that  $\alpha_1 > \alpha_2 > \dots > \alpha_r$ . Then  $\mu$  determines the following chain of (normal) subgroups of  $G$  which ends in  $G$ :

$$(*) \quad {}_\mu G_{\alpha_1} \subset {}_\mu G_{\alpha_2} \subset \dots \subset {}_\mu G_{\alpha_r} = G.$$

A necessary and sufficient condition for two fuzzy (normal) subgroups  $\mu, \eta$  of  $G$  to be equivalent with respect to  $\sim$  has been identified in [27]:  $\mu \sim \eta$  if and only if  $\mu$  and  $\eta$  have the same set of level subgroups, that is they determine the same chain of (normal) subgroups of type  $(*)$ . This result shows that *there exists a bijection between the equivalence classes of fuzzy (normal) subgroups of  $G$  and the set of chains of (normal) subgroups of  $G$  which end in  $G$* . So, the problem of counting all distinct fuzzy (normal) subgroups of  $G$  can be translated into a combinatorial problem on the subgroup lattice  $L(G)$  (or on the normal subgroup lattice  $N(G)$ ) of  $G$ : finding the number of all chains of (normal) subgroups of  $G$  that terminate in  $G$ . Notice also that in our previous papers we have denoted these numbers by  $h(G)$  (respectively by  $h'(G)$ ).

Even for some particular classes of finite groups  $G$ , as finite abelian groups, the problem of determining  $h(G)$  is very difficult. The largest classes of groups for which it was completely solved are constituted by finite cyclic groups (see Corollary 4 of [24]) and by finite elementary abelian  $p$ -groups (see the main result of [23]). Recall that if  $\mathbb{Z}_n$  is the finite cyclic group of order  $n$  and  $n = p_1^{m_1} p_2^{m_2} \dots p_s^{m_s}$  is the decomposition of  $n$  as a product of prime factors, then we have

$$h(\mathbb{Z}_n) = 2^{\sum_{\alpha=1}^s m_\alpha} \sum_{i_2=0}^{m_2} \sum_{i_3=0}^{m_3} \dots \sum_{i_s=0}^{m_s} \left(-\frac{1}{2}\right)^{\sum_{\alpha=2}^s i_\alpha} \prod_{\alpha=2}^s \binom{m_\alpha}{i_\alpha} \binom{m_1 + \sum_{\beta=2}^s (m_\beta - i_\beta)}{m_\alpha},$$

where the above iterated sums are equal to 1 for  $s = 1$ . The number  $h(G)$  has been also computed in several particular situations for dihedral groups and symmetric groups in [18] and [20], respectively. A complete study of finding the number  $h'(G)$  for all above classes of finite groups can be found in [22].

Finally, we recall that an *action* of the group  $G$  on a nonempty set  $X$  is a map  $\rho : X \times G \longrightarrow X$  satisfying the following two conditions:

- a)  $\rho(x, g_1 g_2) = \rho(\rho(x, g_1), g_2)$ , for all  $g_1, g_2 \in G$  and  $x \in X$ ;
- b)  $\rho(x, e) = x$ , for all  $x \in X$ .

Every action  $\rho$  of  $G$  on  $X$  induces an equivalence relation  $R_\rho$  on  $X$ , defined by

$$xR_\rho y \text{ if and only if there exists } g \in G \text{ such that } y = \rho(x, g).$$

The equivalence classes of  $X$  modulo  $R_\rho$  are called the *orbits* of  $X$  relative to the action  $\rho$ . For any  $g \in G$ , we denote by  $Fix_X(g)$  the set of all elements of  $X$  which are fixed by  $g$ , that is

$$Fix_X(g) = \{x \in X \mid \rho(x, g) = x\}.$$

If both  $G$  and  $X$  are finite, then the number of distinct orbits of  $X$  relative to  $\rho$  is given by the equality:

$$N = \frac{1}{|G|} \sum_{g \in G} |Fix_X(g)|.$$

This result is known as the *Burnside's lemma* and plays an important role in solving many problems of combinatorics, finite group theory or graph theory. In the next section it will be also used to compute the number of distinct fuzzy subgroups of a finite group  $G$  with respect to a certain equivalence relation on  $FL(G)$ , induced by an action of the automorphism group  $\text{Aut}(G)$  associated to  $G$  on  $FL(G)$ .

### 3 A new equivalence relation on $FL(G)$

Let  $G$  be a finite group. Then it is well-defined the following action of  $\text{Aut}(G)$  on  $FL(G)$

$$\begin{aligned} \rho : FL(G) \times \text{Aut}(G) &\longrightarrow FL(G), \\ \rho(\mu, f) &= \mu \circ f, \text{ for all } (\mu, f) \in FL(G) \times \text{Aut}(G), \end{aligned}$$

and let us denote by  $\approx_\rho$  the equivalence relation on  $FL(G)$  induced by  $\rho$ , namely

$$\mu \approx_\rho \eta \text{ if and only if there exists } f \in \text{Aut}(G) \text{ such that } \eta = \mu \circ f.$$

As we have seen in Section 2, every fuzzy subgroup of  $G$  determines a chain of subgroups of  $G$  which ends in  $G$  (that is, a chain of type  $(*)$ ). In this way, the above action  $\rho$  can be described in terms of chains of subgroups of  $G$ . Let  $\mu, \eta \in FL(G)$  and put  $\mu(G) = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$  (where  $\alpha_1 > \alpha_2 > \dots > \alpha_m$ ),  $\eta(G) = \{\beta_1, \beta_2, \dots, \beta_n\}$  (where  $\beta_1 > \beta_2 > \dots > \beta_n$ ). If

$$\mathcal{C}_\mu : \quad {}_\mu G_{\alpha_1} \subset {}_\mu G_{\alpha_2} \subset \dots \subset {}_\mu G_{\alpha_m} = G$$

and

$$\mathcal{C}_\eta : \quad {}_\eta G_{\beta_1} \subset {}_\eta G_{\beta_2} \subset \dots \subset {}_\eta G_{\beta_n} = G$$

are the corresponding chains of type  $(*)$ , then we can easily see that  $\mu \approx_\rho \eta$  if and only if  $\text{Im}(\mu) = \text{Im}(\eta)$  (i.e.  $m = n$  and  $\alpha_i = \beta_i$ ,  $i = \overline{1, n}$ ) and there is  $f \in \text{Aut}(G)$  such that  $f({}_\eta G_{\beta_i}) = {}_\mu G_{\alpha_i}$ , for all  $i = \overline{1, n}$ . In other words, two

fuzzy subgroups  $\mu$  and  $\eta$  of  $G$  are equivalent with respect to  $\approx_\rho$  if and only if they have the same image and there is an automorphism  $f$  of  $G$  which maps  $\mathcal{C}_\eta$  into  $\mathcal{C}_\mu$ .

Since we are interested to work only with chains of subgroups, in the following we will consider the equivalence relation  $\approx$  on  $FL(G)$  defined by

$$\mu \approx \eta \text{ iff } \exists f \in \text{Aut}(G) \text{ such that } f(\mathcal{C}_\eta) = \mathcal{C}_\mu.$$

Obviously, this is a little more general as  $\approx_\rho$ : we can easily prove that if  $\mu \approx \eta$ , then their images are not necessarily equal, but certainly there is a bijection between  $Im(\mu)$  and  $Im(\eta)$ . Moreover, we also remark that  $\approx$  generalizes the equivalence relation  $\sim$  (recall that  $\mu \sim \eta \iff \mathcal{C}_\mu = \mathcal{C}_\eta$ , i.e. the above automorphism  $f$  is the identical map of  $G$ ).

Next, we will focus on computing the number  $N$  of distinct fuzzy subgroups of  $G$  with respect to  $\approx$ , that is the number of distinct equivalence classes of  $FL(G)$  modulo  $\approx$ . Denote by  $\overline{\mathcal{C}}$  the set consisting of all chains of subgroups of  $G$  terminated in  $G$ . Then the previous action  $\rho$  of  $\text{Aut}(G)$  on  $FL(G)$  can be seen as an action of  $\text{Aut}(G)$  on  $\overline{\mathcal{C}}$  and  $\approx_\rho$  as the equivalence relation induced by this action. An equivalence relation on  $\overline{\mathcal{C}}$  which is similar with  $\approx$  can also be constructed in the following manner: for two chains

$$\mathcal{C}_1 : H_1 \subset H_2 \subset \dots \subset H_m = G \quad \text{and} \quad \mathcal{C}_2 : K_1 \subset K_2 \subset \dots \subset K_n = G$$

of  $\overline{\mathcal{C}}$ , we put

$$\mathcal{C}_1 \approx \mathcal{C}_2 \text{ iff } m = n \text{ and } \exists f \in \text{Aut}(G) \text{ such that } f(H_i) = K_i, i = \overline{1, n}.$$

In this case the orbit of a chain  $\mathcal{C} \in \overline{\mathcal{C}}$  is  $\{f(\mathcal{C}) \mid f \in \text{Aut}(G)\}$ , while the set of all chains in  $\overline{\mathcal{C}}$  that are fixed by an automorphism  $f$  of  $G$  is  $Fix_{\overline{\mathcal{C}}}(f) = \{\mathcal{C} \in \overline{\mathcal{C}} \mid f(\mathcal{C}) = \mathcal{C}\}$ . Now, the number  $N$  is obtained by applying the Burnside's lemma:

$$(**) \quad N = \frac{1}{|\text{Aut}(G)|} \sum_{f \in \text{Aut}(G)} |Fix_{\overline{\mathcal{C}}}(f)|.$$

Finally, we note that the above formula can successfully be used to calculate  $N$  for any finite group  $G$  whose subgroup lattice  $L(G)$  and automorphism group  $\text{Aut}(G)$  are known.

#### 4 The number of distinct fuzzy subgroups of finite groups

In this section we shall compute explicitly the number  $N$  of all distinct fuzzy subgroups with respect to  $\approx$  for several remarkable classes of finite groups. The comparison with the similar results relative to  $\sim$  obtained in our previous papers is also made.

#### 4.1. The number of distinct fuzzy subgroups of finite cyclic groups

Let  $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$  be the finite cyclic group of order  $n$ . The subgroup structure of  $\mathbb{Z}_n$  is well-known (see [14], I): for every divisor  $d$  of  $n$ , there is a unique subgroup of order  $d$  in  $\mathbb{Z}_n$ , namely  $\langle \frac{n}{d} \rangle$ . Moreover, the following lattice isomorphism holds

$$L(\mathbb{Z}_n) \cong L_n,$$

where  $L_n$  denotes the lattice of all divisors of  $n$ , under the divisibility. It is also well-known the structure of the automorphism group of  $\mathbb{Z}_n$ : if  $f_{\bar{d}} : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  is the map defined by  $f_{\bar{d}}(\bar{k}) = \overline{dk}$ , for all  $\bar{d}, \bar{k} \in \mathbb{Z}_n$ , then

$$\text{Aut}(\mathbb{Z}_n) = \{f_{\bar{d}} \mid (d, n) = 1\}.$$

In particular, we have  $|\text{Aut}(\mathbb{Z}_n)| = \varphi(n)$ , where  $\varphi$  is the Euler's totient function.

In order to find the number  $N$  for  $\mathbb{Z}_n$ , the equality  $(**)$  can be used. Nevertheless, in this case we shall prefer to give a direct solution, founded on the remark that the equivalence relations  $\approx$  and  $\sim$  coincide for a finite cyclic group.

**Theorem 4.1.1.** *For a finite group  $G$ , the following conditions are equivalent:*

- a) *The equivalence relations  $\approx$  and  $\sim$  associated to  $G$  coincide.*
- b)  *$G$  is cyclic.*

**Proof.** Assume first that  $G = \mathbb{Z}_n$  is cyclic. Let  $\mathcal{C}_1, \mathcal{C}_2$  be two chains of subgroups of  $\mathbb{Z}_n$  ended in  $\mathbb{Z}_n$  ( $\mathcal{C}_1 : H_1 \subset H_2 \subset \dots \subset H_p = \mathbb{Z}_n$ ,  $\mathcal{C}_2 : K_1 \subset K_2 \subset \dots \subset K_q = \mathbb{Z}_n$ ) which satisfy  $\mathcal{C}_1 \approx \mathcal{C}_2$  and take  $f \in \text{Aut}(\mathbb{Z}_n)$  such that  $f(\mathcal{C}_1) = \mathcal{C}_2$ . Then  $p = q$  and  $f(H_i) = K_i$ , for any  $i = \overline{1, p}$ . Since  $f$  is an automorphism, the subgroups  $H_i$  and  $K_i$  are of the same order, therefore they must be equal. This shows that  $\mathcal{C}_1 = \mathcal{C}_2$ , that is for a finite cyclic group the equivalence relations  $\approx$  and  $\sim$  are the same.

Conversely, let us assume that for a finite group  $G$  we have  $\approx = \sim$  and take an arbitrary subgroup  $H$  of  $G$ . Then, for any  $x \in G$ , the chains  $H \subset G$  and  $H^x \subset G$  are equivalent modulo  $\approx$ , because there exists the inner automorphism  $f_x$  of  $G$  ( $f_x(g) = xgx^{-1}$ , for all  $g \in G$ ) such that  $f_x(H) = H^x$ . This shows that  $H = H^x$  for all  $x \in G$ , that is  $H$  is a normal subgroup of  $G$ . Thus  $G$  is either a hamiltonian group or an abelian group. In the first case  $G$  is of type  $Q \times \mathbb{Z}_2^k \times A$ , where  $Q = \langle x, y \mid x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1} \rangle$  is the quaternion group of order 8,  $\mathbb{Z}_2^k$  is the direct product of  $k$  copies of  $\mathbb{Z}_2$  and  $A$  is a finite abelian group of odd order. Clearly, the function  $f : G \rightarrow G$  that maps  $x$  into  $y$  and leaves invariant every element of  $\mathbb{Z}_2^k \times A$  is an automorphism of  $G$ . Then the chains  $\langle x \rangle \subset G$  and  $\langle y \rangle \subset G$  are equivalent modulo  $\approx$ , but they are not equal, a contradiction. In the second case  $G$  is a direct product of type

$G_1 \times G_2 \times \cdots \times G_s$ , where each  $G_i$  is an abelian  $p_i$ -group,  $i = \overline{1, s}$ . If we assume that there exists  $i \in \{1, 2, \dots, s\}$  such that the group  $G_i$  is not cyclic, then  $G_i$  possesses two distinct maximal subgroups  $M_i^1, M_i^2$  and an automorphism  $f_i$  satisfying  $f_i(M_i^1) = M_i^2$ . It is easy to see that  $f_i$  can be extended to an automorphism  $f$  of  $G$  mapping  $M_i^1$  into  $M_i^2$ . Again, by our hypothesis, the chains  $M_i^1 \subset G$  and  $M_i^2 \subset G$  must be equal. This leads to  $M_i^1 = M_i^2$ , a contradiction. Therefore every  $G_i$ ,  $i = \overline{1, s}$ , is cyclic and so is  $G$  itself. ■

By Theorem 4.1.1 one obtains that  $N = h(\mathbb{Z}_n)$  and thus the explicit formula of  $h(\mathbb{Z}_n)$  presented in Section 2 can be used to compute  $N$ , too. Hence the following result holds.

**Theorem 4.1.2.** *Let  $n = p_1^{m_1} p_2^{m_2} \dots p_s^{m_s}$  be the decomposition of  $n \in \mathbb{N}, n \geq 2$ , as a product of prime power factors. Then the number  $N$  of all distinct fuzzy subgroups with respect to  $\approx$  of the finite cyclic group  $\mathbb{Z}_n$  is given by the equality*

$$N = 2^{\sum_{\alpha=1}^s m_\alpha} \sum_{i_2=0}^{m_2} \sum_{i_3=0}^{m_3} \dots \sum_{i_s=0}^{m_s} \left(-\frac{1}{2}\right)^{\sum_{\alpha=2}^s i_\alpha} \prod_{\alpha=2}^s \binom{m_\alpha}{i_\alpha} \binom{m_1 + \sum_{\beta=2}^s (m_\beta - i_\beta)}{m_\alpha},$$

where the above iterated sums are equal to 1 for  $s = 1$ .

## 4.2. The number of distinct fuzzy subgroups of finite elementary abelian $p$ -groups

It is well-known (for example, see [14]) that a finite abelian group can be written as a direct product of  $p$ -groups. Therefore the problem of counting the fuzzy subgroups of finite abelian groups must be first solved for  $p$ -groups. A special class of abelian  $p$ -groups is constituted by elementary abelian  $p$ -groups. Such a group  $G$  has a direct decomposition of type

$$\mathbb{Z}_p^n = \underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{n \text{ factors}},$$

where  $p$  is a prime and  $n \in \mathbb{N}^*$ , and its distinct fuzzy subgroups with respect to  $\sim$  have been counted in [23,24]. It is also known that  $\mathbb{Z}_p^n$  possesses a natural linear space structure over the field  $\mathbb{Z}_p$  and that the automorphisms of the group  $\mathbb{Z}_p^n$  coincide with the automorphisms of this linear space. In this way, the automorphism group  $\text{Aut}(\mathbb{Z}_p^n)$  is isomorphic to the general linear group  $GL(n, p)$  and, in particular, we have

$$|\text{Aut}(\mathbb{Z}_p^n)| = \prod_{i=0}^{n-1} (p^n - p^i).$$

Let  $f \in \text{Aut}(\mathbb{Z}_p^n)$  and  $\mathcal{C} \in \text{Fix}_{\overline{\mathcal{C}}}(f)$ , where  $\mathcal{C} : H_1 \subset H_2 \subset \dots \subset H_m = \mathbb{Z}_p^n$ . Then  $f(\mathcal{C}) = \mathcal{C}$ , that is  $f(H_i) = H_i$ , for all  $i = \overline{1, m}$ . This shows that all subspaces  $H_i$ ,  $i = 1, 2, \dots, m$ , of  $\mathbb{Z}_p^n$  are invariant with respect to  $f$ . So, the problem of counting the number of elements of  $\text{Fix}_{\overline{\mathcal{C}}}(f)$  is reduced to a linear algebra problem: determining all chains of subspaces of the linear space  $\mathbb{Z}_p^n$  which ends in  $\mathbb{Z}_p^n$  and are invariant with respect to  $f$ .

Next, we will solve the above problem in the simplest case when  $p = n = 2$  (note that the general case can be treated in a similar manner). The proper subspaces of  $\mathbb{Z}_2^2$ , that in fact correspond to the proper subgroups of the Klein's group, are  $H_1 = \langle(\hat{1}, \hat{0})\rangle$ ,  $H_2 = \langle(\hat{0}, \hat{1})\rangle$  and  $H_3 = \langle(\hat{1}, \hat{1})\rangle$ . As we already have seen, every group automorphism of  $\mathbb{Z}_2^2$  is perfectly determined by a matrix contained in

$$GL(2, 2) = \{A_1 = \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix}, A_2 = \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{1} & \hat{1} \end{pmatrix}, A_3 = \begin{pmatrix} \hat{0} & \hat{1} \\ \hat{1} & \hat{0} \end{pmatrix}, \\ A_4 = \begin{pmatrix} \hat{1} & \hat{1} \\ \hat{0} & \hat{1} \end{pmatrix}, A_5 = \begin{pmatrix} \hat{0} & \hat{1} \\ \hat{1} & \hat{1} \end{pmatrix}, A_6 = \begin{pmatrix} \hat{1} & \hat{1} \\ \hat{1} & \hat{0} \end{pmatrix}\} \cong S_3.$$

For each  $i \in \{1, 2, \dots, 6\}$ , let us denote by  $f_i$  the automorphism induced by the matrix  $A_i$  and by  $L_{f_i}(\mathbb{Z}_2^2)$  the set consisting of all subspaces of  $\mathbb{Z}_2^2$  which are invariant relative to  $f_i$ , i.e.

$$L_{f_i}(\mathbb{Z}_2^2) = \{H \leq \mathbb{Z}_2^2 \mid f_i(H) = H\}.$$

We easily obtain  $L_{f_1}(\mathbb{Z}_2^2) = \{1, H_1, H_2, H_3, \mathbb{Z}_2^2\}$ ,  $L_{f_2}(\mathbb{Z}_2^2) = \{1, H_2, \mathbb{Z}_2^2\}$ ,  $L_{f_3}(\mathbb{Z}_2^2) = \{1, H_3, \mathbb{Z}_2^2\}$ ,  $L_{f_4}(\mathbb{Z}_2^2) = \{1, H_1, \mathbb{Z}_2^2\}$ ,  $L_{f_5}(\mathbb{Z}_2^2) = L_{f_6}(\mathbb{Z}_2^2) = \{1, \mathbb{Z}_2^2\}$ , where 1 denotes the trivial subspace of  $\mathbb{Z}_2^2$ . The above equalities show that  $|Fix_{\overline{\mathcal{C}}}(f_1)| = 8$ ,  $|Fix_{\overline{\mathcal{C}}}(f_i)| = 4$  for  $i = 2, 3, 4$ , and  $|Fix_{\overline{\mathcal{C}}}(f_i)| = 2$  for  $i = 5, 6$ . Clearly, by applying the formula (\*\*), we are now able to compute explicitly the number  $N$  associated to the elementary abelian 2-group  $\mathbb{Z}_2^2$ .

**Theorem 4.2.1.** *The number  $N$  of all distinct fuzzy subgroups with respect to  $\approx$  of the elementary abelian 2-group  $\mathbb{Z}_2^2$  is given by the equality*

$$N = \frac{1}{6} (8 + 3 \cdot 4 + 2 \cdot 2) = 4.$$

**Remark.** By [23,24] we know that  $h(\mathbb{Z}_2^2) = 8$ , i.e. there are 8 distinct chains of subgroups (subspaces) of  $\mathbb{Z}_2^2$  ended in  $\mathbb{Z}_2^2$ , namely  $\mathbb{Z}_2^2$ ,  $1 \subset \mathbb{Z}_2^2$ ,  $H_i \subset \mathbb{Z}_2^2$ ,  $i = 1, 2, 3$ , and  $1 \subset H_i \subset \mathbb{Z}_2^2$ ,  $i = 1, 2, 3$ . From these chains several are equivalent modulo  $\approx$ . More precisely, the distinct equivalence classes with respect to  $\approx$  of the set  $\overline{\mathcal{C}}$  described above are:  $\overline{\mathcal{C}}_1 = \{\mathbb{Z}_2^2\}$ ,  $\overline{\mathcal{C}}_2 = \{1 \subset \mathbb{Z}_2^2\}$ ,  $\overline{\mathcal{C}}_3 = \{H_1 \subset \mathbb{Z}_2^2, H_2 \subset \mathbb{Z}_2^2, H_3 \subset \mathbb{Z}_2^2\}$ ,  $\overline{\mathcal{C}}_4 = \{1 \subset H_1 \subset \mathbb{Z}_2^2, 1 \subset H_2 \subset \mathbb{Z}_2^2, 1 \subset H_3 \subset \mathbb{Z}_2^2\}$ .

### 4.3. The number of distinct fuzzy subgroups of finite dihedral groups

The finite dihedral group  $D_{2n}$  ( $n \geq 2$ ) is the symmetry group of a regular polygon with  $n$  sides and has the order  $2n$ . The most convenient abstract description of  $D_{2n}$  is obtained by using its generators: a rotation  $a$  of order  $n$  and a reflection  $b$  of order 2. Under these notations, we have

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle = \{1, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}.$$

The automorphism group of  $D_{2n}$  is well-known, namely

$$\text{Aut}(D_{2n}) = \{f_{\alpha,\beta} \mid \alpha = \overline{0, n-1} \text{ with } (\alpha, n) = 1, \beta = \overline{0, n-1}\},$$

where  $f_{\alpha,\beta} : D_{2n} \longrightarrow D_{2n}$  is defined by  $f_{\alpha,\beta}(a^i) = a^{\alpha i}$  and  $f_{\alpha,\beta}(a^i b) = a^{\alpha i + \beta} b$ , for all  $i = \overline{0, n-1}$ . We infer that

$$|\text{Aut}(D_{2n})| = n\varphi(n).$$

The structure of the subgroup lattice  $L(D_{2n})$  of  $D_{2n}$  is also well-known: for every divisor  $r$  of  $n$ ,  $D_{2n}$  possesses a subgroup isomorphic to  $\mathbb{Z}_r$ , namely  $H_0^r = \langle a^{\frac{n}{r}} \rangle$ , and  $\frac{n}{r}$  subgroups isomorphic to  $D_r$ , namely  $H_i^r = \langle a^{\frac{n}{r}}, a^{i-1}b \rangle$ ,  $i = 1, 2, \dots, \frac{n}{r}$ .

Next, for each  $f_{\alpha,\beta} \in \text{Aut}(D_{2n})$ , let  $\text{Fix}(f_{\alpha,\beta})$  be the set consisting of all subgroups of  $D_{2n}$  that are invariant relative to  $f_{\alpha,\beta}$ , that is

$$\text{Fix}(f_{\alpha,\beta}) = \{H \leq D_{2n} \mid f_{\alpha,\beta}(H) = H\}.$$

By using some elementary results of group theory, these subsets of  $L(D_{2n})$  can precisely be determined: a subgroup of type  $H_0^r$  belongs to  $\text{Fix}(f_{\alpha,\beta})$  if and only if  $(\alpha, r) = 1$ , while a subgroup of type  $H_i^r$  belongs to  $\text{Fix}(f_{\alpha,\beta})$  if and only if  $(\alpha, r) = 1$  and  $\frac{n}{r}$  divides  $(\alpha - 1)(i - 1) + \beta$ .

Under the above notation, computing  $|\text{Fix}_{\overline{\mathcal{C}}}(f_{\alpha,\beta})|$  is reduced to computing the number of chains of  $L(D_{2n})$  which end in  $D_{2n}$  and are contained in the set  $\text{Fix}(f_{\alpha,\beta})$ . After then, an explicit expression of the number  $N$  associated to the group  $D_{2n}$  will follow from (\*\*).

In the following we will apply this algorithm for the dihedral group  $D_8$ . As we already have seen, the group  $\text{Aut}(D_8)$  consists of  $4\varphi(4) = 4 \cdot 2 = 8$  elements, namely  $\text{Aut}(D_8) = \{f_{1,0}, f_{1,1}, f_{1,2}, f_{1,3}, f_{3,0}, f_{3,1}, f_{3,2}, f_{3,3}\}$ . The corresponding sets of subgroups of  $D_8$  that are invariant relative to the above automorphisms can be described by a direct calculation:  $\text{Fix}(f_{1,0}) = L(D_8)$ ,  $\text{Fix}(f_{1,1}) = \text{Fix}(f_{1,3}) = \text{Fix}(f_{3,1}) = \text{Fix}(f_{3,3}) = \{H_0^1, H_0^2, H_0^4, H_1^4\}$ ,  $\text{Fix}(f_{1,2}) = \{H_0^1, H_0^2, H_0^4, H_1^2, H_1^4\}$ ,  $\text{Fix}(f_{3,0}) = \{H_0^1, H_0^2, H_0^4, H_1^1, H_1^3, H_1^2, H_1^4\}$ ,  $\text{Fix}(f_{3,2}) = \{H_0^1, H_0^2, H_0^4, H_1^1, H_1^3, H_1^2, H_1^4\}$ . Then we easily obtain:  $|\text{Fix}_{\overline{\mathcal{C}}}(f_{1,0})| = 32$ ,  $|\text{Fix}_{\overline{\mathcal{C}}}(f_{1,1})| = |\text{Fix}_{\overline{\mathcal{C}}}(f_{1,3})| = |\text{Fix}_{\overline{\mathcal{C}}}(f_{3,1})| = |\text{Fix}_{\overline{\mathcal{C}}}(f_{3,3})| = 8$ ,  $|\text{Fix}_{\overline{\mathcal{C}}}(f_{1,2})| = 16$ ,  $|\text{Fix}_{\overline{\mathcal{C}}}(f_{3,0})| = |\text{Fix}_{\overline{\mathcal{C}}}(f_{3,2})| = 24$ . Thus the following result holds.

**Theorem 4.3.1.** *The number  $N$  of all distinct fuzzy subgroups with respect to  $\approx$  of the dihedral group  $D_8$  is given by the equality*

$$N = \frac{1}{8} (32 + 4 \cdot 8 + 16 + 2 \cdot 24) = 16.$$

**Remark.** In Theorem 3 of [18] we have proved that the number  $h(D_{2p^m})$  of all distinct fuzzy subgroups relative to  $\sim$  of the dihedral group  $D_{2p^m}$  (where  $p$  is prime and  $m \in \mathbb{N}^*$ ) is

$$h(D_{2p^m}) = \frac{2^m}{p-1}(p^{m+1} + p - 2).$$

In particular, we find  $h(D_{2^m}) = 2^{2m-1}$  and so  $h(D_8) = 32$ , a number which is different from the number  $N$  given by Theorem 4.3.1. We also remark that since  $h(D_8)$  counts all chains of subgroups of  $D_8$  ended in  $D_8$ , it is in fact equal to the number of these chains which are invariant relative to  $f_{1,0} = 1_{D_8}$  (the identical automorphism of  $D_8$ ), that is to  $|Fix_{\overline{\mathcal{C}}}(f_{1,0})|$ .

#### 4.4. The number of distinct fuzzy subgroups of finite symmetric groups

In order to apply the formula (\*\*) for the symmetric group  $S_n$ ,  $n \geq 3$ , we need to know the automorphism group  $\text{Aut}(S_n)$ . An important normal subgroup of this group is the inner automorphism group  $\text{Inn}(S_n)$ , which consists of all automorphisms of  $S_n$  of type  $f_\sigma$ ,  $\sigma \in S_n$ , where  $f_\sigma(\tau) = \sigma\tau\sigma^{-1}$ , for all  $\tau \in S_n$ . On the other hand, we know that if  $n \geq 3$ , then  $\text{Inn}(S_n)$  is isomorphic to  $S_n$ , because the center  $Z(S_n)$  of  $S_n$  is trivial and  $S_n/Z(S_n) \cong \text{Inn}(S_n)$ . According to [14], I, for any  $n \neq 6$  we have

$$\text{Aut}(S_n) = \text{Inn}(S_n) \cong S_n,$$

while for  $n = 6$  we have

$$\text{Aut}(S_6) \cong \text{Aut}(A_6) \text{ and } (\text{Aut}(S_6) : \text{Inn}(S_6)) = 2.$$

In particular, one obtains

$$|\text{Aut}(S_n)| = \begin{cases} n!, & n \neq 6 \\ 2 \cdot 6!, & n = 6. \end{cases}$$

In the following we will focus only on the case  $n \neq 6$ . Then every automorphism of  $S_n$  is of the form  $f_\sigma$  with  $\sigma \in S_n$ . Let  $\mathcal{C} \in \text{Fix}_{\overline{\mathcal{C}}}(f_\sigma)$ , where  $\mathcal{C} : H_1 \subset H_2 \subset \dots \subset H_m = S_n$ . Then  $f_\sigma(\mathcal{C}) = \mathcal{C}$ , that is  $f_\sigma(H_i) = H_i$ , for all  $i = \overline{1, m}$ . This shows that  $H_i^\sigma = H_i$ , i.e.  $\sigma$  is contained in the normalizer  $N_{S_n}(H_i)$  of  $H_i$  in  $S_n$ ,  $i = \overline{1, m}$ . Therefore we have

$$(1) \quad \mathcal{C} \in \text{Fix}_{\overline{\mathcal{C}}}(f_\sigma) \iff \sigma \in N_{S_n}(H_i), \text{ for all } i = \overline{1, m} \iff \sigma \in \bigcap_{i=1}^m N_{S_n}(H_i),$$

which allows us to compute explicitly  $|\text{Fix}_{\overline{\mathcal{C}}}(f_\sigma)|$ . We will exemplify this method for the symmetric group  $S_3$ . It is well-known that  $S_3$  has 6 elements, more precisely

$$S_3 = \{e, \tau_1 = (23), \tau_2 = (13), \tau_3 = (12), \sigma = (123), \sigma^2 = (132)\},$$

and its subgroup lattice consists of the trivial subgroup  $H_0 = \{e\}$ , three subgroups of order 2, namely  $H_i = \langle \tau_i \rangle = \{e, \tau_i\}$ ,  $i = 1, 2, 3$ , a subgroup of

order 3, namely  $H_4 = \langle \sigma \rangle = \{e, \sigma, \sigma^2\}$ , and a subgroup of order 6, namely  $H_5 = S_3$ . Then the chains of subgroups of  $S_3$  ended in  $S_3$  are  $H_5$ ,  $H_i \subset H_5$ ,  $i = \overline{0,4}$ , and  $H_0 \subset H_i \subset H_5$ ,  $i = \overline{1,4}$ . Also, we can easily find the normalizers of all above subgroups:  $N_{S_3}(H_0) = N_{S_3}(H_4) = N_{S_3}(H_5) = S_3$  (in other words  $H_0$ ,  $H_4$  and  $H_5$  are normal in  $S_3$ ), and  $N_{S_3}(H_i) = H_i$ ,  $i = 1, 2, 3$ . By using (1), one obtains  $|Fix_{\overline{\mathcal{C}}}(f_e)| = 10$ ,  $|Fix_{\overline{\mathcal{C}}}(f_{\tau_i})| = 6$ ,  $i = 1, 2, 3$ , and  $|Fix_{\overline{\mathcal{C}}}(f_\sigma)| = |Fix_{\overline{\mathcal{C}}}(f_{\sigma^2})| = 4$ . So, we are able to compute explicitly the number  $N$  associated to the symmetric group  $S_3$  (and also to the dihedral group  $D_6$ , since  $S_3 \cong D_6$ ), in view of the formula (\*\*).

**Theorem 4.4.1.** *The number  $N$  of all distinct fuzzy subgroups with respect to  $\approx$  of the symmetric group  $S_3$  is given by the equality*

$$N = \frac{1}{6} (10 + 3 \cdot 6 + 2 \cdot 4) = 6.$$

We remark that  $N$  is different from  $h(S_3) = 10$  (computed in [20]), as we expected. This is due to the fact that, from the chains of  $\overline{\mathcal{C}}$  described above,  $H_i \subset H_5$ ,  $i = 1, 2, 3$ , are equivalent modulo  $\approx$ , and the same thing can be said about  $H_0 \subset H_i \subset H_5$ ,  $i = 1, 2, 3$ . Finally, we remark that the above reasoning can successfully be applied to compute the number  $N$  associated to the symmetric group  $S_4$ , whose subgroup structure has been completely described in [20], too.

## 5 Conclusions and further research

The study concerning the classification of the fuzzy subgroups of (finite) groups is a significant aspect of fuzzy group theory. It can be made with respect to some natural equivalence relations on the fuzzy subgroup lattices, as the Murali's equivalence relation used in [7]-[11],  $\sim$  used in [16]-[25] and [27], or  $\approx$  used in this paper. Obviously, other such relations can be introduced and investigated. The problem of counting the distinct fuzzy subgroups relative to the above equivalence relations can also be extended to other remarkable classes of finite groups. This will surely constitute the subject of some further research.

Two open problems with respect to this topic are the following.

**Problem 1.** Generalize the results of Section 4, by establishing some explicit formulas for the number of distinct fuzzy subgroups of *arbitrary* elementary abelian  $p$ -groups, dihedral groups and symmetric groups.

**Problem 2.** Classify the fuzzy normal subgroups of a finite group with respect to the equivalence relation  $\approx$  defined in Section 3. Use the particular classes of finite groups studied in [22], whose normal subgroup structure can completely be described.

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